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K-quasi-additive fuzzy integrals of set-valued mappings*

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Abstract We first define the quasi-addition and quasi-multiplication operations by introducing the inductive operator, and then, in the K-quasi-additive fuzzy measure space, we establish the K-quasi-additive fuzzy integral of a generally measurable set-valued mapping. Applying the integral transformation theorem, some basic properties of the K-quasi-additive fuzzy integrals with respect to this kind of set-valued mapping are studied. Finally, the generalized monotone convergence theorems of this kind of fuzzy integrals are obtained.

Keywords: set-valued mapping, inductive operator, integrable selections, K-quasi-additive fuzzy integrals.

Set-valued integral is a new theory, which was developed with the applications of economics, control theory, optimization, non-smooth analysis and statistics, etc., in the later period of the 1960s. Inspired by the problems in economics, Aumann[1] defined the integral of the set-valued mapping on R^n according to the single-valued Lebesgue's integrable selection of the measurable set-valued mapping in 1965. At the middle period of the 1970s, Datko[2], Artstein[3] and Hiai et al. generalized many theories in R^n to the Banach space, discussed the set-valued condition expectation and the existence of the set-valued martingales, and gave the integral expressions of the bounded integrable set-valued mappings. From the 1980s to the 1990s, Sugeno et al.[4] and Zhang et al.[5,6] conducted much good work on the theory and applications of set-valued measure, stochastic set and set-valued stochastic process, respectively, especially in the research area of the integrals of fuzzy set-valued mappings.

With the emergence and development of fuzzy measures and fuzzy integrals, establishment of the fuzzy integral of the set-valued function is considered. However, a fuzzy measure does not satisfy general additivity, which makes the fuzzy integral not linear. Thus, such defined set-valued integral is very difficult to study. In a more common non-continuous fuzzy measure space, for the measurable set-valued mapping, Jang[7] defined the fuzzy Choquet integral with respect to the set-valued mapping in 1997, and gave some basic properties of this kind of fuzzy Choquet integrals. Furthermore, some expressions for fuzzy Choquet integrals of set-valued mappings were given in Ref. [8]. In this paper, based on our work in Refs. [9—12], the K-quasi-additive fuzzy integral of the set-valued mapping in the K-quasi-additive fuzzy measure space is established for the first time, and some elementary properties of this kind of K-quasi-additive fuzzy integrals are discussed by applying the integral transformation theorem. Finally, we give the generalized monotone convergence theorems of the K-quasi-additive fuzzy integrals.

1 Elementary definitions

Let X be a given classical set, $R^+=[0,+\infty)$, $\mathfrak{R}$ be a σ-algebra consisting of some subsets in X, $(X,\mathfrak{R})$ denote an arbitrarily given measurable space, and $P(R^+)$ denote a set of all the power sets on R^+.

Definition 1.1. Let $K: R^+\rightarrow R^+$ be a strictly increasing continuous function. If it satisfies the following conditions (1) $K(0)=0$, $K(1)=1$; (2) $\lim_{x\rightarrow+\infty} K(x)=+\infty$, then the function K is called an inductive operator defined on R^+.

Clearly, the converse operator K^{-1} exists, and K^{-1} is still an inductive operator. Of course, K^{-1} is also continuous and increasing, and $K(K^{-1}(x))=K^{-1}(K(x))=x$ for all $x\in R^+$. For example, we may choose $K(x)=\ln[1+(e-1)x]$, or $K(x)=x$ for any $x\in R^+$, then $K^{-1}(x)=(e^x-1)/(e-1)$ or

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$K^{-1}(x)=x$ for every $x\in R^+$. Obviously, K and K^{-1} are inductive operators.

Definition 1.2. Let K be a given inductive operator on R^+, for all a, $b\in R^+$. Define the K-quasi sum "$\oplus$" and K-quasi product "$\otimes$" of a and b as follows: $a\oplus b=K^{-1}(K(a)+K(b))$, $a\otimes b=K^{-1}(K(a)K(b))$, for any a, b, c, $d\in[0,+\infty)$. Consequently we can obtain the following conclusions:

(1) $(a\oplus b)\oplus c=a\oplus(b\oplus c)$, $(a\otimes b)\otimes c=a\otimes(b\otimes c)$;

(2) $a\oplus b=b\oplus a$, $a\otimes b=b\otimes a$, $a\oplus 0=a$, $a\otimes 0=0$, $a\otimes 1=a$;

(3) $a\otimes(b\oplus c)=(a\otimes b)\oplus(a\otimes c)$;

(4) if $a\leqslant b$, $c\leqslant d$, then $a\oplus c\leqslant b\oplus d$ and $a\otimes c\leqslant b\otimes d$;

(5) $K(a\oplus b)=K(a)+K(b)$ and $K(a\otimes b)=K(a)\cdot K(b)$;

(6) $K^{-1}(a+b)=K^{-1}(a)\oplus K^{-1}(b)$ and $K^{-1}(a\cdot b)=K^{-1}(a)\otimes K^{-1}(b)$;

(7) if $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}\subset[0,+\infty)$, and $\lim_{n\to\infty}a_n=a$, $\lim_{n\to\infty}b_n=b$, then $\lim_{n\to\infty}(a_n\oplus b_n)=a\oplus b$ and $\lim_{n\to\infty}(a_n\otimes b_n)=a\otimes b$.

Definition 1.3. Let $(X,\Re)$ be an arbitrary measurable space, K be a given inductive operator, $\hat{\mu}:\Re\to[0,+\infty]$ be a set-function, if the following conditions (1)—(4) are satisfied:

(1) $\hat{\mu}(\phi)=0$;

(2) if A, $B\in\Re$, and $A\cap B=\phi$, then $\hat{\mu}(A\cup B)=\hat{\mu}(A)\oplus\hat{\mu}(B)$;

(3) if $\{A_n\}_{n=1}^{\infty}\subset\Re$, and $A_n\uparrow A$, then $\hat{\mu}(A_n)\uparrow\hat{\mu}(A)$;

(4) if $\{A_n\}_{n=1}^{\infty}\subset\Re$, $A_n\downarrow A$, and there exists a natural number n_0 satisfying $\hat{\mu}(A_{n_0})<+\infty\Rightarrow\hat{\mu}(A_n)\downarrow\hat{\mu}(A)$, then $\hat{\mu}$ is called a K-quasi-additive fuzzy measure, and the corresponding triplet $(X,\Re,\hat{\mu})$ is called a K-quasi-additive fuzzy measure space.

Definition 1.4.[9,13] Let $(X,\Re,\hat{\mu})$ be a given K-quasi-additive fuzzy measure space, f be a nonnegative real valued measurable function on $(X,\Re)$, K be an inductive operator, $D=\{A_1,A_2,\cdots,A_n\}$ be an arbitrary finite measurable partition on X. Let $S_K(f,D,A)=\oplus\sum_{i=1}^{n}[\sup_{x\in A_i\cap A}f(x)\otimes\hat{\mu}(A_i\cap A)]$, and $\int_A^{(K)}f\mathrm{d}\hat{\mu}=\sup_D S_K(f,D,A)$. Then $\int_A^{(K)}f\mathrm{d}\hat{\mu}$ is called a K-quasi-additive fuzzy integral of f with respect to $\hat{\mu}$ on A. If $\sup_D S_K(f,D,A)<+\infty$, then f is called $K-\hat{\mu}$ integrable on A. The set of all the $K-\hat{\mu}$ integrable functions on A are denoted as $L_A^1(\hat{\mu})$.

Lemma 1.[13] Let $(X,\Re,\hat{\mu})$ be a K-quasi-additive fuzzy measure space, K be an inductive operator, and $\mu(\cdot)=K(\hat{\mu}(\cdot))$. Then set-function μ is a classical Lebesgue measure.

Lemma 2.[9] **(Transformation Theorem of Integral)** Let $(X,\Re,\hat{\mu})$ be a K-quasi-additive fuzzy measure space, f be a nonnegative real valued measurable function on $(X,\Re)$, K be an inductive operator, $A\in\Re$. Then $\int_A^{(K)}f\mathrm{d}\hat{\mu}=K^{-1}\left[\int_A K\circ f\mathrm{d}\mu\right]$, and $\int_A^{(K)}f\mathrm{d}\hat{\mu}$ exists if and only if its composite function $K\circ f$ is Lebesgue integrable on A. Here $\mu(\cdot)=K(\hat{\mu}(\cdot))$, $\int_A K\circ f\mathrm{d}\mu$ is a classical Lebesgue integral, and μ is a classical Lebesgue measure.

Definition 1.5.[7] Let $(X,\Re)$ be a measurable space, $F:X\to P(R^+)-\{\phi\}$ be a set-valued mapping. If its graph is measurable, i.e. $Gr(F)=\{(x,y)\in X\times R^+\mid y\in F(x)\}\in\Re\times B(R^+)$, then F is called a measurable set-valued mapping on X, where $B(R^+)$ is a Borel field on nonnegative real number set R^+.

Definition 1.6. Let $(X,\Re,\hat{\mu})$ be a K-quasi-additive fuzzy measure space, $F:X\to P(R^+)-\{\phi\}$ be a measurable set-valued mapping, $A\in\Re$. Define $(K)\int_A F\mathrm{d}\hat{\mu}:=\left\{\int_A^{(K)}f\mathrm{d}\hat{\mu}\mid f\in S_A(F)\right\}$. Then $(K)\int_A F\mathrm{d}\hat{\mu}$ is called a K-quasi-additive fuzzy integral of set-valued mapping F on A, where K is an inductive operator, $S_A(F)=\{f\in L_A^1(\hat{\mu})\mid f(x)\in F(x)\ a.e.\ \text{to}\ A\}$, i.e. $S_A(F)$ denote all of the integrable selections of F on A. If $(K)\int_A F\mathrm{d}\hat{\mu}\neq\phi$, then F is called K-quasi-integrable on A.

Remark 1. Obviously, by Definition 1.6, Lemma 2 and Definition 1.4, we know that f is $K-\hat{\mu}$ integrable on A if $f \in S_A(F)$, which implies $K^{-1}\left[\int_A K \circ f d\mu\right] < +\infty$ or $\int_A K \circ f d\mu < +\infty$.

Definition 1.7.[7] Let $(X, \Re, \hat{\mu})$ be a K-quasi-additive fuzzy measure space, $A \in \Re$, a set-valued mapping F be K-quasi-integrable on A. If there exists an integrable function $h \in L_A^1(\hat{\mu})$ such that the norm $\| F(x) \| := \sup_{\omega \in F(x)} |\omega| \leqslant h(x)$ for all $x \in X$, then F is called K-quasi-bounded integrable.

Definition 1.8.[7] Let real number sets A, $B \in P(R^+) - \{\phi\}$, if the following conditions (1) and (2) are fulfilled: (1) for each $x_0 \in A$, there exists a $y_0 \in B$ such that $x_0 \leqslant y_0$; (2) for each $y_0 \in B$, there exists an $x_0 \in A$ such that $x_0 \leqslant y_0$. Then we call A is weaker than B, writing $A \prec B$.

Definition 1.9. Let real number sets A, $B \in P(R^+) - \{\phi\}$. Define the generalized quasi-sum, quasi-multiplication and number multiply operation of sets A and B, respectively:

$$A \oplus B := \{x \oplus y \mid x \in A, y \in B\},$$
$$A \otimes B := \{x \otimes y \mid x \in A, y \in B\},$$
$$k \otimes A := \{k \otimes x \mid x \in A\}$$

for arbitrary constant $k \geqslant 0$.

Definition 1.10. Let $(X, \Re)$ be a measurable space, F and G be measurable set-valued mappings on X. Denote their extension operations as follows:

(1) $(F \cup G)(x) = F(x) \cup G(x)$, $(F \cap G)(x) = F(x) \cap G(x)$ for all $x \in X$;

(2) $(F \oplus G)(x) = F(x) \oplus G(x)$, $(F \otimes G)(x) = F(x) \otimes G(x)$ for all $x \in X$;

(3) $(\alpha \otimes F)(x) = \alpha \otimes F(x)$ $\forall x \in X$ and for any constant $\alpha \in R^+$.

2 Elementary properties

In this paper, for simplicity, as for the given K-quasi-additive fuzzy measure space $(X, \Re, \hat{\mu})$, we always suppose K is a given inductive operator, and $S_A(F)$ denote all of the integrable selections of set-valued mapping F on A.

Theorem 2.1. Let $(X, \Re, \hat{\mu})$ be a given K-quasi-additive fuzzy measure space, F and G be measurable set-valued mappings, which are K-quasi-integrable on $A \in \Re$. Then $F \cup G$, $F \cap G$, and $F \oplus G$ are K-quasi-integrable on A. And if the composite function $K \circ f$ and $K \circ g$ are square integrable for any $f \in S_A(F)$ and $g \in S_A(G)$, then set-valued mapping $F \otimes G$ is K-quasi-integrable on A, too.

Proof. (1) We first prove that $F \cup G$ is K-quasi-integrable on A.

In fact, let $(K)\int_A F d\hat{\mu} \neq \phi$ and $(K)\int_A G d\hat{\mu} \neq \phi$. Thus, there certainly exist the integrable selections $f \in S_A(F)$ and $g \in S_A(G)$ on F such that $f(x) \in F(x)$ $a.e.$ to A and $g(x) \in G(x)$ $a.e.$ to A, respectively. In addition we can obtain

$$x_0 = \int_A^{(K)} f d\hat{\mu} = K^{-1}\left[\int_A K \circ f d\mu\right] < +\infty$$

and

$$y_0 = \int_A^{(K)} g d\hat{\mu} = K^{-1}\left[\int_A K \circ g d\mu\right] < +\infty.$$

Let $h(x) = f(x) \vee g(x)$, for all $x \in A$. Then $h(x) \in F(x) \cup G(x) = (F \cup G)(x)$ $a.e.$ to A, and K^{-1} is strictly increasing. At the same time, we have

$$\begin{aligned}\int_A^{(K)} h d\hat{\mu} &= K^{-1}\left[\int_A K \circ (f \vee g) d\mu\right] \\ &= K^{-1}\left[\int_A (K \circ f) \vee (K \circ g) d\mu\right] \\ &\leqslant K^{-1}\left[\int_A (K \circ f + K \circ g) d\mu\right] \\ &= K^{-1}\left[\int_A K \circ f d\mu + \int_A K \circ g d\mu\right] \\ &= K^{-1}\left[\int_A K \circ f d\mu\right] \oplus K^{-1}[K \circ g d\mu] \\ &< +\infty.\end{aligned}$$

Therefore, h is an integrable selection on $F \cup G$.

Consequently, $z_0 = \int_A^{(K)} h d\hat{\mu} \in (K)\int_A (F \cup G) d\hat{\mu} \neq \phi$. By Definition 1.6, we know that set-valued mapping $F \cup G$ is K-quasi-integrable on A.

(2) Similarly, let $h(x) = f(x) \wedge g(x)$ for each $x \in A$, we can prove that $F \cap G$ is K-quasi-integrable.

(3) Let $m(x) = f(x) \oplus g(x)$ for every $x \in A$, then $m(x) \in F(x) \oplus G(x) = (F \oplus G)(x)$ $a.e.$ to A, and we have

$$\begin{aligned}\int_A^{(K)} m d\hat{\mu} &= K^{-1}\left[\int_A K \circ (f \oplus g) d\mu\right] \\ &= K^{-1}\left[\int_A (K \circ f + K \circ g) d\mu\right]\end{aligned}$$

$$= K^{-1}\left[\int_A K\circ f\mathrm{d}\mu\right]\oplus K^{-1}\left[\int_A K\circ g\mathrm{d}\mu\right]$$
$$< +\infty.$$

Hence, m is an integrable selection on set-valued mapping $F\oplus G$. Let $z_0' = \int_A^{(K)} m\,\mathrm{d}\hat{\mu}$. Then

$$z_0' \in (K)\int_A (F\oplus G)\mathrm{d}\hat{\mu} \neq \phi.$$

Consequently, $F\oplus G$ is K-quasi-integrable on A.

(4) Let $\psi(x) = f(x)\otimes g(x)$ for any $x\in A$. Then $\psi(x)\in F(x)\otimes G(x) = (F\otimes G)(x)$ $a.e.$ to A, because $K\circ f$ and $K\circ g$ are square integrable, that is $\int_A (K\circ f)^2\mathrm{d}\mu < +\infty$, $\int_A (K\circ g)^2\mathrm{d}\mu < +\infty$ and K^{-1} is strictly increasing. According to Lemma 2 and Schwarz's inequality, we obtain

$$\begin{aligned}\int_A^{(K)} m\,\mathrm{d}\hat{\mu} &= K^{-1}\left[\int_A K\circ (f\otimes g)\mathrm{d}\mu\right]\\ &= K^{-1}[(K\circ f)(K\circ g)\mathrm{d}\mu]\\ &\leqslant K^{-1}\left[\left(\int_A (K\circ f)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\right.\\ &\quad\left.\cdot\left(\int_A (K\circ g)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\right]\\ &= K^{-1}\left[\left(\int_A (K\circ f)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\right]\\ &\quad\otimes K^{-1}\left[\left(\int_A (K\circ g)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\right]\\ &< +\infty.\end{aligned}$$

Thus, m is an integrable selection on $F\otimes G$, and $z_0'' := \int_A^{(K)} m\,\mathrm{d}\hat{\mu} \in (K)\int_A (F\otimes G)\mathrm{d}\hat{\mu} \neq \phi$. In this case, the set-valued mapping $F\otimes G$ is K-quasi-integrable on A.

Theorem 2.2. Let $(X, \Re, \hat{\mu})$ be a given K-quasi-additive fuzzy measure space, a set-valued mapping F be K-quasi-integrable, and $A, B\in\Re$ with $A\subset B$. Then $(K)\int_A F\mathrm{d}\hat{\mu} \prec (K)\int_B F\mathrm{d}\hat{\mu}$.

Proof. Let $x_0\in (K)\int_A F\mathrm{d}\hat{\mu}\neq\phi$. Then there exists an integrable selection $f\in S_A(F)$ such that

$$x_0 = \int_A^{(K)} f\mathrm{d}\hat{\mu} = K^{-1}\left[\int_A K\circ f\mathrm{d}\mu\right] < +\infty,$$

because $A\subset B$, considering the monotonicity of Lebesgue's integrals and the strictly increasing property of inductive operator K^{-1}, we can obtain

$$x_0 = K^{-1}\left[\int_A K\circ f\mathrm{d}\mu\right] \leqslant K^{-1}\left[\int_B K\circ f\mathrm{d}\mu\right]$$
$$= \int_B^{(K)} f\mathrm{d}\hat{\mu} \in (K)\int_B F\mathrm{d}\hat{\mu}.$$

Let $y_0 = K^{-1}\left[\int_B K\circ f\mathrm{d}\mu\right]$. Then they satisfy $x_0\leqslant y_0\in (K)\int_B F\mathrm{d}\hat{\mu}$.

Using the similar method, we may prove that for any $y_0\in (K)\int_B F\mathrm{d}\hat{\mu}\neq\phi$, there exists an $x_0\in (K)\int_A F\mathrm{d}\hat{\mu}$ such that $x_0\leqslant y_0$. Thus, by Definition 1.8, we get $(K)\int_A F\mathrm{d}\hat{\mu} \prec (K)\int_B F\mathrm{d}\hat{\mu}$.

Theorem 2.3. Let $(X, \Re, \hat{\mu})$ be a K-quasi-additive fuzzy measure space, the closed set-valued mapping F and G be K-quasi-bounded integrable, and $F\prec G$, $A\in\Re$. Then $(K)\int_A F\mathrm{d}\hat{\mu} \prec (K)\int_A G\mathrm{d}\hat{\mu}$.

Proof. Let $x_0\in (K)\int_A F\mathrm{d}\hat{\mu}\neq\phi$. Then there exists an integrable selection $f\in S_A(F)$ such that

$$x_0 = \int_A^{(K)} f\mathrm{d}\hat{\mu} = K^{-1}\left[\int_A K\circ f\mathrm{d}\mu\right] < +\infty,$$

and $f(x)\in F(x)$ $a.e.$ to A.

Because $F(x)\prec G(x)$ for every $x\in X$, let $g(x) = \sup\{\omega \mid \omega\in G(x),\ \omega\geqslant f(x)\}$.

Since $G(x)$ is a closed set, we have $g(x)\in G(x)$ and $g(x)\geqslant f(x)$. G is bounded integrable, so there exists a $K-\hat{\mu}$ integrable function $h\in L_A^1(\hat{\mu})$ such that

$$\| G(x)\| = \sup_{\omega\in G(x)} |\omega| \leqslant h(x).$$

Then from $\{\omega \mid \omega\in G(x),\ \omega\geqslant f(x)\}\subset\{\omega\mid\omega\in G(x)\}$, we get

$$\begin{aligned}g(x) &= \sup_{\omega\in G(x),\,\omega\geqslant f(x)} \omega \leqslant \sup_{\omega\in G(x),\,\omega\geqslant f(x)} |\omega|\\ &\leqslant \sup_{\omega\in G(x)} |\omega| = \|G(x)\| \leqslant h(x).\end{aligned}$$

Therefore, the function g is $K-\hat{\mu}$ integrable, i.e. $g\in S_A(G)$. Let $y_0 = K^{-1}\left[\int_A K\circ g\mathrm{d}\mu\right]$. Then we have $x_0 = K^{-1}\left[\int_A K\circ f\mathrm{d}\mu\right] \leqslant K^{-1}\left[\int_A K\circ g\mathrm{d}\mu\right] = y_0 = \int_A^{(K)} g\mathrm{d}\hat{\mu}\in (K)\int_A G\mathrm{d}\hat{\mu}$. On the other hand, for any $y_0\in (K)\int_A G\mathrm{d}\hat{\mu}\neq\phi$, there exists an integrable selection $g\in S_A(G)$ such that $y_0 = K^{-1}\left[\int_A K\circ g\mathrm{d}\mu\right]$.

From $F(x)\prec G(x)$ for all $x\in X$, let

$$f(x) = \inf\{\omega \mid \omega\in F(x), \omega\leqslant g(x)\}.$$

Because $F(x)$ is a closed set and bounded integrable, similar to the above proof, we can obtain $f(x) \leqslant g(x)$ and the function f is $K-\hat{\mu}$ integrable, too. Thus $f \in S_A(F)$ and

$$y_0 = K^{-1}\left[\int_A K \circ g \mathrm{d}\mu\right] \geqslant K^{-1}\left[\int_A K \circ f \mathrm{d}\mu\right] = \int_A^{(K)} f \mathrm{d}\hat{\mu} := x_0 \in (K)\int_A F \mathrm{d}\hat{\mu}.$$

Hence, by Definition 1.8, we obtain

$$(K)\int_A F \mathrm{d}\hat{\mu} \prec (K)\int_A G \mathrm{d}\hat{\mu}.$$

Theorem 2.4. Let $(X, \mathfrak{R}, \hat{\mu})$ be K-quasi-additive fuzzy measure space, if inductive operator K satisfies $K(a)K\left(\frac{1}{a}\right) = 1$, and the set-valued mapping F is K-quasi-integrable. Then for every constant $a \geqslant 0$ and $A \in \mathfrak{R}$, $(K)\int_A a \otimes F \mathrm{d}\hat{\mu} = a \otimes (K)\int_A F \mathrm{d}\hat{\mu}$.

Proof. Obviously, the conclusion holds whenever $a = 0$. Without loss of generality, let $a > 0$. Actually, on the one hand, for every $x_0 \in (K)\int_A a \otimes F \mathrm{d}\hat{\mu} \neq \phi$, there exists an integrable selection $f \in S_A(a \otimes F)$ on $a \otimes F$ such that

$$x_0 = \int_A^{(K)} f \mathrm{d}\hat{\mu} = K^{-1}\left[\int_A K \circ f \mathrm{d}\mu\right] < +\infty.$$

Thus $f(x) \in (a \otimes F)(x) = a \otimes F(x)$ $a.e.$ to A.

Let $g(x) = \frac{1}{a} \otimes f(x)$ for all $x \in X$. Applying Definition 1.2 and the hypothesis condition, we obtain

$$\begin{aligned} a \otimes g(x) &= a \otimes \frac{1}{a} \otimes f(x) \\ &= K^{-1}\left(K(a)K\left(\frac{1}{a}\right)\right) \otimes f(x) \\ &= K^{-1}(1) \otimes f(x) = 1 \otimes f(x) \\ &= f(x). \end{aligned}$$

Therefore, $f(x) = a \otimes g(x) \in a \otimes F(x)$ $a.e.$ to A, and $g(x) \in F(x)$ $a.e.$ to A.

By Definition 1.2 (5) and (6), we can obtain

$$\begin{aligned} K^{-1}\left[\int_A K \circ g \mathrm{d}\mu\right] &= K^{-1}\left[\int_A K\left(\frac{1}{a} \otimes f(x)\right) \mathrm{d}\mu\right] \\ &= K^{-1}\left[\int_A K\left(\frac{1}{a}\right) K(f(x))\right] \mathrm{d}\mu \\ &= K^{-1}\left[K\left(\frac{1}{a}\right)\int_A K \circ f \mathrm{d}\mu\right] \\ &= \frac{1}{a} \otimes K^{-1}\left[\int_A K \circ f \mathrm{d}\mu\right] \\ &< +\infty. \end{aligned}$$

Hence, the function g is $K-\hat{\mu}$ integrable, thus g is an integrable selection of F, i.e. $g \in S_A(F)$, and $\int_A^{(K)} g \mathrm{d}\hat{\mu} \in (K)\int_A F \mathrm{d}\hat{\mu}$. In this case, we have

$$\begin{aligned} x_0 &= K^{-1}\left[\int_A K \circ f \mathrm{d}\mu\right] \\ &= K^{-1}\left[\int_A K(a \otimes g(x)) \mathrm{d}\mu\right] \\ &= K^{-1}\left[\int_A K(a) K(g(x)) \mathrm{d}\mu\right] \\ &= K^{-1}\left[K(a)\int_A K \circ g \mathrm{d}\mu\right] \\ &= K^{-1}[K(a)] \otimes K^{-1}\left[\int_A K \circ g \mathrm{d}\mu\right] \\ &= a \otimes \int_A^{(K)} g \mathrm{d}\hat{\mu} \in a \otimes (K)\int_A F \mathrm{d}\hat{\mu}. \end{aligned}$$

Consequently,

$$(K)\int_A a \otimes F \mathrm{d}\hat{\mu} \subset a \otimes (K)\int_A F \mathrm{d}\hat{\mu}.$$

On the other hand, for any $y_0 \in a \otimes (K)\int_A F \mathrm{d}\hat{\mu} \neq \phi$, we have $\frac{1}{a} \otimes y_0 \in (K)\int_A F \mathrm{d}\hat{\mu}$. Then there exists an integrable selection $f \in S_A(F)$ on F such that

$$\frac{1}{a} \otimes y_0 = \int_A^{(K)} f \mathrm{d}\hat{\mu} = K^{-1}\left[\int_A K \circ f \mathrm{d}\mu\right] < +\infty.$$

Let $g(x) = a \otimes f(x)$ for all $x \in X$, $f(x) \in F(x)$ $a.e.$ to A and $f \in L_A^1(\hat{\mu})$.

Therefore, $g(x) \in a \otimes F(x) = (a \otimes F)(x)$ $a.e.$ to A.

Similar to the above proof, we can easily prove $g \in S_A(a \otimes F)$, and

$$\begin{aligned} y_0 &= a \otimes \int_A^{(K)} f \mathrm{d}\mu = a \otimes K^{-1}\left[\int_A K \circ f \mathrm{d}\mu\right] \\ &= K^{-1}\left[K(a)\int_A K \circ f \mathrm{d}\mu\right] \\ &= K^{-1}\left[\int_A K(a)(K(f(x))) \mathrm{d}\mu\right] \\ &= K^{-1}\left[\int_A K(a \otimes f(x)) \mathrm{d}\mu\right] \\ &= \int_A^{(K)} a \otimes f \mathrm{d}\hat{\mu} \in (K)\int_A a \otimes F \mathrm{d}\hat{\mu}. \end{aligned}$$

Consequently, $a \otimes (K)\int_A F \mathrm{d}\hat{\mu} \subset (K)\int_A a \otimes F \mathrm{d}\hat{\mu}$. Furthermore, we have

$$(K)\int_A a \otimes F \mathrm{d}\hat{\mu} = a \otimes (K)\int_A F \mathrm{d}\hat{\mu}.$$

Theorem 2.5. Let $(X, \mathfrak{R}, \hat{\mu})$ be a K-quasi-additive fuzzy measure space, a set-valued mapping F be K-quasi-integrable, A, $B\in\mathfrak{R}$. Then

$$(K)\int_{A\cup B}F\mathrm{d}\hat{\mu}\prec(K)\int_A F\mathrm{d}\hat{\mu}\oplus(K)\int_B F\mathrm{d}\hat{\mu}.$$

Proof. Let $x_0\in(K)\int_{A\cup B}F\mathrm{d}\hat{\mu}\neq\phi$. Then there exists an integrable selection $f\in S_A(F)$ such that $x_0=\int_{A\cup B}^{(K)}f\mathrm{d}\hat{\mu}$. As $A\cup B=(A-B)\cup B$ and $(A-B)\cap B=\phi$, $A-B\subset A$.

From the Lebesgue's integral property and the increasing of K^{-1}, we can derive

$$\begin{aligned}x_0&=K^{-1}\Big[\int_{A\cup B}K\circ f\mathrm{d}\mu\Big]\\&=K^{-1}\Big[\int_{A-B}K\circ f\mathrm{d}\mu+\int_B K\circ f\mathrm{d}\mu\Big]\\&\leqslant K^{-1}\Big[\int_A K\circ f\mathrm{d}\mu\Big]\oplus K^{-1}\Big[\int_B K\circ f\mathrm{d}\mu\Big].\end{aligned}$$

Let $y_0:=K^{-1}\Big[\int_A K\circ f\mathrm{d}\mu\Big]$, $z_0:=K^{-1}\Big[\int_B K\circ f\mathrm{d}\mu\Big]$. Then

$$\begin{aligned}x_0&\leqslant y_0\oplus z_0\\&=\int_A^{(K)}f\mathrm{d}\hat{\mu}\oplus\int_B^{(K)}f\mathrm{d}\hat{\mu}\in(K)\int_A F\mathrm{d}\hat{\mu}\\&\quad\oplus\int_B F\mathrm{d}\hat{\mu}.\end{aligned}$$

On the contrary, for every $\omega_0\in(K)\int_A F\mathrm{d}\hat{\mu}\oplus(K)\int_B F\mathrm{d}\hat{\mu}$, there certainly exist the integrable selection $f\in S_A(F)$ and $g\in S_B(F)$ such that

$$\omega_0=K^{-1}\Big[\int_A K\circ f\mathrm{d}\mu\Big]\oplus K^{-1}\Big[\int_B K\circ g\mathrm{d}\mu\Big].$$

Let $h(x)=f(x)\wedge g(x)$, for all $x\in X$, from Theorem 2.1 (2), we can prove that function h is an integrable selection of F. Furthermore, we can obtain

$$\begin{aligned}\omega_0&=K^{-1}\Big[\int_A K\circ f\mathrm{d}\mu+\int_B K\circ g\mathrm{d}\mu\Big]\\&\geqslant K^{-1}\Big[\int_{A-B}K\circ h\mathrm{d}\mu+\int_B K\circ h\mathrm{d}\mu\Big]\\&=K^{-1}\Big[\int_{A\cup B}K\circ h\mathrm{d}\mu\Big].\end{aligned}$$

Let $x_0:=K^{-1}\Big[\int_{A\cup B}K\circ h\Big]$, clearly, $\omega_0\geqslant x_0$ and $x_0=\int_{A\cup B}^{(K)}h\mathrm{d}\hat{\mu}\in(K)\int_{A\cup B}F\mathrm{d}\hat{\mu}$.

By Definition 1.8, we have $(K)\int_{A\cup B}F\mathrm{d}\hat{\mu}\prec(K)\int_A F\mathrm{d}\hat{\mu}\oplus(K)\int_B F\mathrm{d}\hat{\mu}$.

Theorem 2.6. Let $(X,\mathfrak{R},\hat{\mu})$ be a K-quasi-additive fuzzy measure space, the set-valued mapping F and G be K-quasi-integrable, $A\in\mathfrak{R}$. Then

(1) $(K)\int_A(F\cup G)\mathrm{d}\hat{\mu}=(K)\int_A F\mathrm{d}\hat{\mu}\cup(K)\int_A G\mathrm{d}\hat{\mu}$;

(2) $(K)\int_A(F\cap G)\mathrm{d}\hat{\mu}=(K)\int_A F\mathrm{d}\hat{\mu}\cap(K)\int_A G\mathrm{d}\hat{\mu}$.

Proof. On the one hand, for any $x_0\in(K)\int_A(F\cup G)\mathrm{d}\hat{\mu}\neq\phi$, there exists an integrable selection $m\in S_A(F\cup G)$ on $F\cup G$ such that

$$x_0=\int_A^{(K)}m\mathrm{d}\hat{\mu}=K^{-1}\Big[\int_A K\circ m\mathrm{d}\mu\Big]<+\infty.$$

Here, the function $m\in L_A^1(\mu)$ and $m(x)\in(F\cup G)(x)=F(x)\cup G(x)$ *a.e.* to A, which follows that $m(x)\in F(x)$ *a.e.* to A or $m(x)\in G(x)$ *a.e.* to A. Thus, m is an integrable selection of F or G, i.e. $m\in S_A(F)\cup S_A(G)$, which means that $x_0\in(K)\int_A F\mathrm{d}\hat{\mu}\cup(K)\int_A G\mathrm{d}\hat{\mu}$.

Consequently,

$$(K)\int_A(F\cup G)\mathrm{d}\hat{\mu}\subset(K)\int_A F\mathrm{d}\hat{\mu}\cup(K)\int_A G\mathrm{d}\hat{\mu}.$$

On the other hand, for each $x_0\in(K)\int_A F\mathrm{d}\hat{\mu}\cup(K)\int_A G\mathrm{d}\hat{\mu}$, without loss of generality, we let $x_0\in(K)\int_A F\mathrm{d}\hat{\mu}$. Then there exists an integrable selection $f\in S_A(F)$ such that $x_0=(K)\int_A f\mathrm{d}\hat{\mu}$ and $f\in L_A^1(\hat{\mu})$, i.e. $f(x)\in F(x)$ *a.e.* to A. In addition, $f(x)\in F(x)\subset F(x)\cup G(x)=(F\cup G)(x)$ *a.e.* to A, that is, $f\in S_A(F\cup G)$ and $x_0\in(K)\int_A(F\cup G)\mathrm{d}\hat{\mu}$.

Therefore,

$$(K)\int_A F\mathrm{d}\hat{\mu}\cup(K)\int_A G\mathrm{d}\hat{\mu}\subset(K)\int_A(F\cup G)\mathrm{d}\hat{\mu},$$

Thus, we can obtain

$$(K)\int_A(F\cup G)\mathrm{d}\hat{\mu}=(K)\int_A F\mathrm{d}\hat{\mu}\cup(K)\int_A G\mathrm{d}\hat{\mu}.$$

We can use the similar method to prove (2), so omit it.

Theorem 2.7. Let $(X,\mathfrak{R},\hat{\mu})$ be a K-quasi-additive fuzzy measure space, the set-valued mapping F

and G be K-quasi-integrable, $A \in \mathfrak{R}$. Then

$$(K)\int_A (F \oplus G)\mathrm{d}\hat{\mu} = (K)\int_A F\mathrm{d}\hat{\mu} \oplus (K)\int_A G\mathrm{d}\hat{\mu}.$$

Proof. In fact, for any $x_0 \in (K)\int_A (F \oplus G)\mathrm{d}\hat{\mu} \neq \phi$, there exists an integrable selection $h \in S_A(F \otimes G)$ such that $x_0 = \int_A^{(K)} h\mathrm{d}\hat{\mu} = K^{-1}\left[\int_A K \circ h \mathrm{d}\mu\right] < +\infty$ and $h(x) \in (F \oplus G)(x) = F(x) \oplus G(x)$ $a.e.$ to A. Hence, there exist $y_x \in F(x)$ and $z_x \in G(x)$ such that $h(x) = y_x \oplus z_x$ $a.e.$ to A. Thus

$$\begin{aligned} x_0 &= K^{-1}\left[\int_A K(y_x \oplus z_x)\mathrm{d}\mu\right] \\ &= K^{-1}\left[\int_A (K(y_x) + K(z_x))\mathrm{d}\mu\right] \\ &= K^{-1}\left[\int_A K(y_x)\mathrm{d}\mu\right] \oplus K^{-1}\left[\int_A K(z_x)\mathrm{d}\mu\right] \\ &= \int_A^{(K)} y_x \mathrm{d}\hat{\mu} \oplus \int_A^{(K)} z_x \mathrm{d}\hat{\mu} \in (K)\int_A F\mathrm{d}\hat{\mu} \\ &\oplus K\int_A G\mathrm{d}\hat{\mu}. \end{aligned}$$

Using the similar method, from Theorem 2.5, we may prove that the anti-inclusion is tenable.

Hence,

$$(K)\int_A (F \oplus G)\mathrm{d}\hat{\mu} = (K)\int_A F\mathrm{d}\hat{\mu} \oplus (K)\int_A G\mathrm{d}\hat{\mu}$$

holds.

3 Generalized monotone convergence theorem

In this section, we first give the monotonity definitions of the sequence of sets and the set-valued mapping, and then, we discuss that the corresponding K-quasi-additive fuzzy integrals satisfy monotonity whenever the sequence of the K-quasi-integrable set-valued mappings satisfies monotonity, i.e. the generalized convergence theorems hold.

Definition 3.1.[8] Let a sequence of sets $\{A_n\}_{n=1}^{\infty} \subset P(R^+)$, $A = \{x \mid x = \lim_{k\to\infty} x_{n_k}, x_{n_k} \in A_{n_k}\}$, if $A_n \subset A_{n+1}$, $n = 1, 2, \cdots$. Then $\{A_n\}_{n=1}^{\infty}$ is called monotone increasing convergent to A, simply denoted as $A_n \uparrow A$. Let $A = \{x \mid x = \lim_{n\to\infty} x_n, x_n \in A_n\}$, if $A_{n+1} \subset A_n$, $n = 1, 2, \cdots$. Then $\{A_n\}_{n=1}^{\infty}$ is called monotone decreasing convergent to A, simply denoted as $A_n \downarrow A$.

Definition 3.2.[8] Let $(X, \mathfrak{R}, \hat{\mu})$ be a K-quasi-additive fuzzy measure space, $A \in \mathfrak{R}$, the sequence of measurable set-valued mappings $\{F_n\}_{n=1}^{\infty}$ be K-quasi-integrable, and the set-valued mapping F be K-quasi-integrable, too. If (1) $S_A(F_n) \subset S_A(F_{n+1})$, $n = 1, 2, \cdots$; (2) for any $f \in S_A(F)$, there always exists a monotone increasing integrable selection subsequence $f_{n_k} \in S_A(F_{n_k})$ $(k = 1, 2, \cdots)$ such that $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$. Then $\{F_n\}_{n=1}^{\infty}$ is called monotone increasing convergent to F, written as $F_n \uparrow F$. If (3) $S_A(F_{n+1}) \subset S_A(F_n)$, $n = 1, 2, \cdots$; (4) for each $f \in S_A(F)$, there always exists a monotone decreasing integrable selection sequence $f_n \in S_A(F_n)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$. Then $\{F_n\}_{n=1}^{\infty}$ is called monotone decreasing convergent to F, written as $F_n \downarrow F$.

Theorem 3.1. Let $(X, \mathfrak{R}, \hat{\mu})$ be a K-quasi-additive fuzzy measure space, $A \in \mathfrak{R}$, the sequence of measurable set-valued mappings $\{F_n\}_{n=1}^{\infty}$ be K-quasi-integrable, a measurable set-valued mapping F be K-quasi-integrable, too. If $F_n \uparrow F$, then

$$(K)\int_A F_n \mathrm{d}\hat{\mu} \uparrow (K)\int_A F\mathrm{d}\hat{\mu}.$$

Proof. For an arbitrary certain natural number n, let $y \in (K)\int_A F_n\mathrm{d}\hat{\mu} := A_n$. Then there exists an integrable selection $g \in S_A(F_n)$ such that

$$y = \int_A^{(K)} g\mathrm{d}\hat{\mu} = K^{-1}\left[\int_A K \circ g\mathrm{d}\mu\right] < +\infty.$$

Because $S_A(F_n) \subset S_A(F_{n+1})$, $n = 1, 2, \cdots$, it is easy to know $g \in S_A(F_{n+1})$ and $y = \int_A^{(K)} g\mathrm{d}\hat{\mu} \in (K)\int_A F_{n+1}\mathrm{d}\hat{\mu}$. Therefore, $A_n = (K)\int_A F_n\mathrm{d}\hat{\mu} \subset (K)\int_A F_{n+1}\mathrm{d}\hat{\mu} = A_{n+1}$.

On the other hand, for any $y \in (K)\int_A F\mathrm{d}\hat{\mu} \neq \phi$, there exists $h \in S_A(F)$ such that $y = \int_A^{(K)} h\mathrm{d}\hat{\mu}$.

From $F_n \uparrow F$ and Definition 4.2, we can know that there certainly exists a monotone increasing integrable selection subsequence $h_{n_k} \in S_A(F_{n_k})$ $(k = 1, 2, \cdots,)$ such that $\lim_{k\to\infty} h_{n_k}(x) = h(x)$ for all $x \in A$.

Consider the coutinuity and monotone increasing property of inductive operators K^{-1} and K. From the Monotone Convergence Theorem of Lebesgue's integrals, we can obtain

$$\begin{aligned}\lim_{k\to\infty} y_{n_k} &= \lim_{k\to\infty}\int_A^{(K)} h_{n_k}\mathrm{d}\hat{\mu} \\ &= \lim_{k\to\infty} K^{-1}\Big[\int_A K\circ h_{n_k}\mathrm{d}\mu\Big] \\ &= K^{-1}\Big[\lim_{k\to\infty}\int_A K\circ h_{n_k}\mathrm{d}\mu\Big] \\ &= K^{-1}\Big[\int_A K(\lim_{k\to\infty} h_{n_k})\mathrm{d}\mu\Big] \\ &= K^{-1}\Big[\int_A K\circ h\mathrm{d}\mu\Big] = \int_A^{(K)} h\mathrm{d}\hat{\mu} = y.\end{aligned}$$

Obviously, $\int_A^{(K)} h_{n_k}\mathrm{d}\hat{\mu}\in(K)\int_A F_{n_k}\mathrm{d}\hat{\mu}$. By Definition 3.1, we can obtain $(K)\int_A F_n\mathrm{d}\hat{\mu}\uparrow(K)\int_A F\mathrm{d}\hat{\mu}$.

Theorem 3.2. Let $(X,\mathfrak{R},\hat{\mu})$ be a K-quasi-additive fuzzy measure space, $A\in\mathfrak{R}$, the sequence of measurable set-valued mappings $\{F_n\}_{n=1}^{\infty}$ be K-quasi-integrable, a measurable set-valued mapping F be K-quasi-integrable, too. If $F_n\downarrow F$, then

$$(K)\int_A F_n\mathrm{d}\hat{\mu}\downarrow(K)\int_A F\mathrm{d}\hat{\mu}.$$

Proof. First, similar to the proof of Theorem 4.1, it is obvious that

$$A_{n+1} = (K)\int_A F_{n+1}\mathrm{d}\hat{\mu}\subset(K)\int_A F_n\mathrm{d}\hat{\mu} = A_n.$$

Second, for every $\forall y\in(K)\int_A F\mathrm{d}\hat{\mu}\neq\phi$, there exists an integrable selection $f\in S_A(F)$ such that

$$y = \int_A^{(K)} f\mathrm{d}\hat{\mu} = K^{-1}\Big[\int_A K\circ f\mathrm{d}\mu\Big] < +\infty.$$

Because $F_n\downarrow F$, by Definition 3.2, there exists a decreasing integrable selection sequence $f_n\in S_A(F_n)$ $(n=1,2,\cdots,)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x\in A$.

Therefore, we can obtain

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty}\int_A^{(K)} f_n\mathrm{d}\hat{\mu} = \lim_{n\to\infty} K^{-1}\Big[\int_A K\circ f_n\mathrm{d}\mu\Big] \\ &= K^{-1}\Big[\lim_{n\to\infty}\int_A K\circ f_n\mathrm{d}\mu\Big] \\ &= K^{-1}\Big[\int_A K(\lim_{n\to\infty} f_n)\mathrm{d}\mu\Big] \\ &= K^{-1}\Big[\int_A K\circ f\mathrm{d}\mu\Big] = \int_A^{(K)} f\mathrm{d}\hat{\mu} = y.\end{aligned}$$

On the other hand, it is obvious that

$$\int_A^{(K)} f_n\mathrm{d}\hat{\mu}\in(K)\int_A F_n\mathrm{d}\hat{\mu}.$$

Thus, from Definition 3.1, $(K)\int_A F_n\mathrm{d}\hat{\mu}\downarrow(K)\int_A F\mathrm{d}\hat{\mu}$ holds.

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